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Letter to the Editor

# A combined equivalent linearization and averaging perturbation method for non-linear oscillator equations

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In recent years a new class of nonlinear oscillator differential equation has been investigated. While they take the form

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad 0 < \varepsilon \ll 1, \tag{1a}$$

$$g(-x) = -g(x), \tag{1b}$$

the elastic force term $g(x)$ may be non-polynomial and/or contain no linear hook-law linear term [1–3], and $\varepsilon$ is a small positive parameter. Such expressions may also occur in the energy generation/dissipative function $f(x, \dot{x})$. For this class of second order differential equation the standard perturbation procedures cannot be applied since the $\varepsilon = 0$ limit does not give a harmonic oscillator equation and it is on this assumption that all the standard methods are based. The following are particular examples of equation belonging to the class defined by Eq. (1):

$$\ddot{x} + x^3 = \varepsilon(1 - x^2)(\dot{x})^{1/3}, \tag{2a}$$

$$\ddot{x} + x^{1/3} + \varepsilon x^3 = 0, \tag{2b}$$

$$\ddot{x} + x^{1/3} = \varepsilon(1 - x^2)\dot{x}. \tag{2c}$$

It should be indicated that the method of harmonic balance can be used to construct analytical approximations to the periodic solutions of the systems given in Eqs. (1) and (2). However, this procedure does not allow the determination of the transient behavior of the solutions for the case where limit cycles exist [3].

The main purpose of this note is to propose a perturbation method that can be used to construct first order (in $\varepsilon$) solutions to Eq. (1) for the case where limit cycles exist. This method combines the important features of two standard perturbation procedures: equivalent linearization [4] and first order averaging [3,4]. A strength of the method is that it permits a

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direct calculation of the transient behaviors for the solutions as they approach the limit points or limit cycles. Since for many problems only the general qualitative features of the oscillatory behavior are needed, the proposed method can provide this information along with (first order in $\varepsilon$) estimates for the angular frequency and the limit-cycles amplitudes.

To proceed, consider Eq. (1) with the initial conditions

$$x(0) = A = \text{given}, \quad \dot{x}(0) = 0. \tag{3}$$

A linearization of the function $g(x)$ can be done by expanding $f(A\cos\theta)$, where $\theta = \omega(A)t$ [4]. Doing this gives

$$g(x) = g(A\cos\theta) = g_0(A)A\cos\theta + (\text{higher order harmonics}). \tag{4}$$

The linearization process consists in dropping all the higher order harmonic terms and replacing $A\cos\theta$ in the first term on the right by $x$, i.e.,

$$g(x) \to g_0(A)A\cos\theta \to g_0(A)x. \tag{5}$$

Note that given $g(x)$, the coefficient $g_0(A)$ can be directly calculated. If the angular frequency is defined as

$$[\omega(A)]^2 = g_0(A) \tag{6}$$

and Eqs. (5) and (6) are substituted into Eq. (1), the following expression results :

$$\ddot{x} + [\omega(A)]^2 x = \varepsilon f(x, \dot{x}), \quad 0 < \varepsilon \ll 1, \tag{7}$$

This last equation is now of the form for which the first order averaging methods [3,4,5] can be applied, subject to the initial conditions of Eq. (3).

The first order averaging method consists of assuming that Eq. (7) has a solution of the form [3–5]

$$x(t,\varepsilon) = a(t,\varepsilon)\cos[\omega t + \phi(t,\varepsilon)], \tag{8a}$$

$$\dot{x}(t,\varepsilon) = -a(t,\varepsilon)\omega\sin[\omega t + \phi(t,\varepsilon)], \tag{8b}$$

where the solutions to the following equations provide a first order estimate for the amplitude $a(t,\varepsilon)$ and phase $\phi(t,\varepsilon)$ functions:

$$\frac{da}{dt} = -\left(\frac{\varepsilon}{2\pi\omega}\right)\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi, \tag{9a}$$

$$\frac{d\phi}{dt} = -\left(\frac{\varepsilon}{2\pi a\omega}\right)\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi \tag{9b}$$

with the requirement that $a(0,\varepsilon) = A$. Note that since this is a first order in $\varepsilon$ calculation, in general the initial conditions, given by Eq. (3), are only satisfied to order $\varepsilon$. For the above procedure, it follows that

$$x(0) = A, \quad \dot{x}(0) = O(\varepsilon). \tag{10}$$

The initial details of this method will be illustrated by applying it to two examples.

First, consider a Duffings oscillator with linear damping,

$$\ddot{x} + x^3 = -\varepsilon\dot{x} \tag{11}$$

corresponding to $g(x) = x^3$ and $f(x, \dot{x}) = -\dot{x}$. Now

$$g(A\cos\theta) = A^3(\cos\theta)^3 = \left(\frac{3A^2}{4}\right)A\cos\theta + \text{HOH} \tag{12}$$

and the linearization of $g(x)$ is

$$g(x) \to \left(\frac{3A^2}{4}\right)x \quad \text{and} \quad [\omega(A)]^2 = \frac{3A^2}{4}. \tag{13}$$

Therefore, Eq. (11) is transformed to

$$\ddot{x} + [\omega(A)]^2 x = -\varepsilon\dot{x} \tag{14}$$

and Eq. (9) are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left(\frac{\varepsilon}{2}\right)a, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \tag{15}$$

with solutions

$$a(t,\varepsilon) = A\exp\left(-\frac{\varepsilon t}{2}\right), \quad \phi(t,\varepsilon) = \phi_0 = \text{constant}. \tag{16}$$

Consequently, an approximation to the solution of Eq. (11) is

$$x(t,\varepsilon) = A\mathrm{e}^{-(\varepsilon t/2)}\cos\left[\sqrt{\tfrac{3}{4}}\,t\right], \tag{17}$$

where $\phi_0$ has been selected to be zero.

It should be observed that the following non-linear oscillatory problem [6] with linear damping can be also solved using this technique:

$$\ddot{x} + x^{1/3} = -\varepsilon\dot{x}. \tag{18}$$

Using the result [7]

$$(\cos\theta)^{1/3} = a_1\cos\theta + a_2\cos(3\theta) + a_3\cos(5\theta) + \cdots, \tag{19}$$

where $a_1 = 1.15960$, the linearization of Eq. (18) gives

$$\ddot{x} + [\omega(A)]^2 x = -\varepsilon\dot{x} \quad \text{with } [\omega(A)]^2 = \frac{a_1}{A^{2/3}} \tag{20}$$

and the approximate solution is

$$x(t,\varepsilon) = A\mathrm{e}^{-(\varepsilon t/2)}\cos[\omega(A)t]. \tag{21}$$

Consider now the equations [1,6]

$$\ddot{x} + x^{3/5} = \varepsilon(1 - x^2)\dot{x} \tag{22}$$

and

$$\ddot{x} + x^{1/3} = \varepsilon(1 - x^2)\dot{x} \tag{23}$$

or the more general equation which may have limit cycles

$$\ddot{x} + g(x) = \varepsilon F(x^2, \dot{x}^2)\dot{x}, \tag{24}$$

$$g(x) = x^{(2m+1)/(2n+1)}, \quad (m, n) \text{ integers}. \tag{25}$$

The proposed combined equivalent linearization and averaging perturbation method is carried out in the following steps:

(1) Replace $g(x)$ by its equivalent linear form

$$\begin{aligned} x^{\alpha} &\to [A^{(\alpha-1)} b_1]x, \quad \alpha = \frac{2m+1}{2n+1}, \\ &= [\omega(A)]^2 x, \end{aligned} \tag{26}$$

where

$$x^{\alpha} \to (A\cos\theta)^{\alpha} = A^{\alpha}[b_1 \cos\theta + b_2 \cos 3\theta + \cdots] \tag{27}$$

was used. (Note that the coefficients $(b_1, b_2, \ldots)$ can be explicitly calculated [7].) The corresponding differential equation is

$$\ddot{x} + [\omega(A)]^2 x = \varepsilon F(x^2, \dot{x}^2)\dot{x}. \tag{28}$$

(2) Now use the method of first order averaging to determine the values of possible limit-cycle amplitudes. This can be done by using Eq. (9a) with the function $f$ replaced by the function $F$ from Eq. (24), i.e.,

$$G(a) = \int_0^{2\pi} F(a^2 \cos^2\psi, \omega^2 a^2 \sin^2\psi)(-\omega a \sin\psi)\sin\psi \, d\psi = 0. \tag{29}$$

The real and positive zeros of $G(a) = 0$ correspond to the limit-cycle amplitudes. However, for many systems modelling natural phenomena, only a single such root exists, i.e., the limit cycle is unique. Calling this value $a = \bar{a}$, the calculation proceeds by replacing $\omega(A)$ by $\omega(\bar{a})$, i.e., Eq. (28) becomes

$$\ddot{x} + [\omega(\bar{a})]^2 x = \varepsilon F(x^2, \dot{x}^2)\dot{x}. \tag{30}$$

(3) The method of first order averaging is now applied to Eq. (30) with the initial conditions

$$a(0, \varepsilon) = a_0, \quad \phi(0, \varepsilon) = \phi_0, \tag{31}$$

where $a_0$ and $\phi_0$ are "arbitrary" constants.

The following example gives a good illustration of this general procedure [1]:

$$\ddot{x} + x^3 = \varepsilon(1 - x^2)\dot{x}. \tag{32}$$

The linearization of the $g(x) = x^3$ term gives

$$\ddot{x} + [\omega(A)]^2 x = \varepsilon(1 - x^2)\dot{x}, \quad [\omega(A)]^2 = \frac{3A^2}{4}. \tag{33}$$

Further, the evaluation of the amplitude differential Eq. (9a) leads to the result

$$G(a) = a\left[1 - \frac{a^2}{4}\right] = 0. \tag{34}$$

Inspection of this equation shows that $\bar{a} = 2$. Consequently, first order averaging is now to be applied to

$$\ddot{x} + 3x = \varepsilon(1 - x^2)\dot{x}, \tag{35}$$

where $[\omega(2)]^2 = 3$ has been used in Eq. (33). Doing the appropriate calculations gives the results

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{\varepsilon a}{2}\right)\left[1 - \frac{a^2}{4}\right], \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \tag{36}$$

with solutions

$$a(t, \varepsilon) = \frac{2a_0}{[a_0^2 + (4 - a_0^2)\mathrm{e}^{-\varepsilon t}]^{1/2}}, \quad \phi(t, \varepsilon) = \phi_0. \tag{37}$$

Note that

$$a(t, \varepsilon) \xrightarrow[t \text{ large}]{} 2 \tag{38}$$

and the approximate solution to Eq. (32) is

$$x(t, \varepsilon) = a(t, \varepsilon)\cos(\sqrt{3}t + \phi_0) \tag{39}$$

with

$$a(t, \varepsilon) \xrightarrow[t \text{ large}]{} 2\cos\left(\sqrt{3}t + \phi_0\right). \tag{40}$$

This latter result is in agreement with the previous results of Mickens [1], i.e., Eq. (32) has a limit-cycle solution such that, for $0 < \varepsilon \ll 1$, the limiting amplitude has a value close to $a = 2$, and the angular frequency is $\omega = \sqrt{3}$. A similar calculation can be done for Eq. (23).

In summary, we have constructed a new perturbation procedure which allows the easy and direct estimation of the limit-cycle parameters for oscillatory systems for which the usual perturbation methods cannot be applied. The power of the procedure was demonstrated by using it to obtain approximate analytical solutions to three non-linear equations for which the elastic force is not only non-linear but also contains no linear terms. A major advantage of the procedure is that it allows the calculation of the transient behavior to the limit cycle.

Finally, it should be indicated that one possible way to improve on the results presented here is to obtain the exact angular frequency for the left-side terms in Eq. (1), i.e.,

$$\ddot{y} + g(y) = 0. \tag{41}$$

Since this corresponds to a conservative oscillator, a first integral can be easily gotten; it is given by

$$\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + H(y) = E, \tag{42a}$$

where

$$H(y) = \int^y g(z)\,\mathrm{d}z. \tag{42b}$$

For initial conditions, $y(0) = A$ and $\dot{y}(0) = 0$, the period is given by

$$T(A) = \frac{2\pi}{\omega(A)} = 4\int_A^0 \frac{\mathrm{d}y}{\sqrt{2[H(A) - H(y)]}}. \tag{43}$$

This exact value for $\omega(A)$ can then be used in Eq. (26).

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